Dynamic programming

Ass. (u_t) – vector of control variables
(x_t) – " – state variables

$(x_t), (u_t) \in \mathbb{R}^T$ when T – finite
or e.g. $(x_t), (u_t) \in \ell_\infty$ when T – infinite
[the question remains, how to define the distance/norm in the infinite dimensional space of sequences]

State variables are governed by the law of motion
$$x_{t+1} = m_t(x_t, u_t)$$

Ass. Decision-maker maximizes $W_t = \sum_{s=t}^{T-1} f_s(x_s, u_s)$

or (in infinite horizon) $W_t = \sum_{s=t}^{\infty} f_s(x_s, u_s)$,
with T and x_0 given, x_T given

Additional restrictions: $(x_t, u_t) \in C_t$.

Def. Continuation sequence $u_{t,T-1} = \{u_s : s = t, \dots, T-1\}$
x_t and $u_{t,T-1}$ imply a continuation sequence of x's:
$$x_{t,T-1} = \{x_s : s = t+1, \dots, T\}.$$
Let $z_{t,T} = \{u_{t,T-1} \cup x_{t+1,T}\}$
By $\Phi(x_t)$ we denote the set of ~~admissible~~ admissible $z_{t,T}$.

Note: $z_{t,T}\big|_a^b$ — finite subsequence from a to b.

Then the decision maker max $W_t : \mathbb{R}^{2(T-t)} \to \mathbb{R}$
$$W(z_{t,T}) = \sum_{s=t}^{T-1} f_s(x_s, u_s).$$

Optimization problems

Key assumptions

- additive separability of the objective function
- f_t, m_t depend only on current values of variables
- x_{t+1} depends only on 1-period lagged variables.

!

Def. Value function

$$V(x_t, t, x_T, T) = \max_{u_{t,T-1}} \Big\{ W[z_{t,T}] = \cancel{s.t.\ x_s}$$

$$= \sum_{s=t}^{T-1} f_s(u_s, x_s) \text{ s.t. } x_{s+1} = m_s(u_s, x_s) \text{ given, } t, T, x_t, x_T \text{ given,}$$

$$(x_s, u_s) \in C_s \subseteq \mathbb{R}^{n+m} \text{ for each } s \Big\}$$

GENERAL CASE, FOR A WHILE THINK OF $C_s \subseteq \mathbb{R}^2$

~~for~~

Does the value function exist?

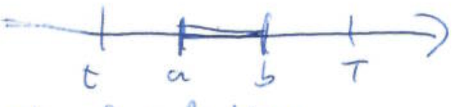
If T - finite, then $W: \mathbb{R}^{2(T-1)} \to \mathbb{R}$.

If f_s - continuous functions then W - continuous.

If $x_{s+1} = m_s(u_s, x_s)$ — the set of (finite sequences of) admissible variables is an intersection of closed sets.

If C_s are bounded, then the admissible set is compact and the existence of max follows from Weierstrass.

Theorem (Principle of optimality)



Let $z^*_{t,T} = \{u^*_{t,T-1}, x^*_{t,T}\}$ be the optimal solution.

Then the optimal solution to

$$V(x^*_a, a, x^*_b, b) = \max_{u_{a,b-1}} \Big\{ W(z_{a,b-1}) \text{ s.t. } x_{s+1} = m_s(x_s, u_s), \; a, b, x^*_a, x^*_b \text{ given, } (x_s, u_s) \in C^s \; \forall s \Big\}$$

is given by $z^*_{t,T}\big|_a^{b-1}$.

Intuition – each portion of the optimal plan is optimal in its own right.

Implication: the optimal solution is TIME CONSISTENT.
We may approach the problem sequentially, breaking up the dynamic problem into a sequence of static ones.

Proof (by contradiction)

Suppose $z^*_{t,T}\big|_a^{b-1}$ is not optimal. Then there exists $z'_{a,b-1}$ such that $W(z'_{a,b-1}) > W(z^*_{t,T}\big|_a^{b-1})$. By the time additivity of the objective fct,

$$W\left(z^*_{t,T}\big|_t^{a-1}\right) + W\left(z'_{a,b-1}\right) + W\left(z^*_{t,T}\big|_b^{T-1}\right) > W\left(z^*_{t,T}\big|_t^{T-1}\right).$$

We have found a feasible solution $\left\{ z^*_{t,T}\big|_t^{a-1} \cup z'_{a,b-1} \cup z^*_{t,T}\big|_b^{T-1} \right\}$ which is better than $z^*_{t,T}$, contradicting its optimality. □

How to split the problem?

$$W(z_{t,T}) = \sum_{s=t}^{T-1} f_s(u_s, x_s) = f_t(u_t, x_t) + \sum_{s=t+1}^{T-1} f_s(u_s, x_s) =$$

$$= f_t(u_t, x_t) + W(z_{t+1,T}).$$

Hence

$$V(x_t, t; x_T, T) = \max_{u_t, u_{t+1,T-1}} \left\{ f_t(u_t, x_t) + W(z_{t+1,T}) \right\} =$$

$$= \max_{u_t} \left\{ f_t(u_t, x_t) + \max_{u_{t+1,T-1}} \left\{ W[z_{t+1,T}) \right\} \right\}$$

Def. The Bellman equation

$$V(x_t, t; x_T, T) = \max_{u_t} \left\{ f_t(u_t, x_t) + V(x_{t+1}, t+1; x_T, T) \quad \text{s.t. } x_{t+1} = m_t(x_t, u_t) \right\}$$

Fact. The Bellman equation is recursive.

Def. The policy function $u_t^*(x_t)$

$$u_t^*(x_t) = \arg\max_{u_t} \left\{ f_t(x_t, u_t) + V(x_{t+1}, t+1; x_T, T) \text{ s.t. } x_{t+1} = m_t(u_t, x_t) \right\}.$$

Fact. The Bellman equation is a functional equation in the unknown function V.

Why is it useful?

↳ finite $T \Rightarrow$ backward induction

↳ infinite $T \Rightarrow$ value function iteration

$\Rightarrow$ Euler equation approach

$\Downarrow$ Blackwell theorem verifies existence of V in a large class of (discounted) problems.

Backward induction

x_T – known, u_T – trivial

$\Downarrow$

$x_T = m_{T-1}(u_{T-1}, x_{T-1})$, solve for u_{T-1} given x_{T-1}

$\Downarrow$

$x_{T-1} = m_{T-2}(u_{T-2}, x_{T-2})$, solve for u_{T-2} given x_{T-2}

$\Downarrow$

all the way back to t.

- works only when there is a finite horizon T.

Example (backward induction)

$$\max_{c_{0,2}} \sum_{t=0}^{2} \ln c_t \quad \text{s.t.} \quad s_{t+1} = (s_t - c_t)(1+r), \ r>0$$

s_0 - given, $s_t \geq 0$ for all t, $c_t \leq s_t$.

Bellman eq's.

$$V(s_0, 0; s_2, 2) = \max_{c_0} \{ \ln c_0 + V(s_1; 1; s_2; 2) \}$$

$$V(s_1, 1; s_2, 2) = \max_{c_1} \{ \ln c_1 + V(s_2; 2; s_2, 2) \}$$

$$V(s_2, 2; s_2, 2) = \max_{c_2} \{ \ln c_2 \}.$$

Solve backwards:

at $t=2$: $\max_{c_2} \{\ln c_2\}$ s.t. $c_2 \leq s_2$. Hence $c_2^*(s_2) = s_2$.

This implies the value $V(s_2, 2; s_2, 2) = \ln s_2 = \ln[(s_1 - c_1)(1+r)]$

At $t=1$: $\max_{c_1} \{ \ln c_1 + \ln((s_1 - c_1)(1+r)) \}$.

$$\frac{\partial}{\partial c_1}: \ \frac{1}{c_1} + \frac{-1}{s_1 - c_1} = 0 \Rightarrow c_1 = s_1 - c_1 \Rightarrow c_1 = \frac{1}{2} s_1.$$

Policy fct: $c_1^*(s_1) = \frac{1}{2} s_1$. Value fct $V(s_1, 1; s_2; 2) =$

$= \ln(\frac{1}{2} s_1) + \ln(\frac{1}{2} s_1) + \ln(1+r)$.

At $t=0$: $\max_{c_0} \{ \ln c_0 + 2\ln\frac{1}{2}(s_0 - c_0)(1+r) + \ln(1+r) \} =$

$$= \max_{c_0} \{ \ln c_0 + 2\ln\tfrac{1}{2}(s_0 - c_0) + 3\ln(1+r) \}$$

$$\frac{\partial}{\partial c_0}: \ \frac{1}{c_0} + 2\frac{-1}{s_0 - c_0} = 0 \Rightarrow c_0 = \frac{s_0 - c_0}{2} \Rightarrow c_0 = \frac{1}{3} s_0.$$

Policy fct: $c_0^*(s_0) = \frac{1}{3} s_0$.

Value fct $V(s_0, 0; s_2, 2) = \ln\frac{1}{3}s_0 + 2\ln\frac{1}{2}s_1 + \ln(1+r) =$

$= \ln\frac{1}{3}s_0 + 2\ln\frac{1}{3}s_0 + 3\ln(1+r) = 3\ln(\frac{1}{3}s_0) + 3\ln(1+r)$.

Optimal sequence satisfies: $c_0^*(s_0) = \frac{1}{3}s_0$, $c_1 = \frac{1}{3}s_0(1+r)$, $c_2 = \frac{1}{3}s_0(1+r)^2$

s_0; $s_1 = \frac{2}{3}s_0(1+r)$, $s_2 = \frac{1}{3}s_0(1+r)^2$.

Theorem (Envelope theorem)

Let $F(\alpha) = \max_{x \in X} f(x, \alpha)$, where f-differentiable, X-open

Then $\frac{d}{d\alpha} F(\alpha) = \frac{\partial f}{\partial \alpha}(x, \alpha)$.

Proof.

$$\frac{d}{d\alpha} F(\alpha) = \frac{d}{d\alpha} f(\overbrace{x^*(\alpha)}^{x^*(\alpha) = \arg\max_{x \in X} f(x,\alpha)}, \alpha) = \underbrace{\frac{\partial f}{\partial x}(x^*(\alpha), \alpha)}_{0} \cdot \frac{\partial x^*(\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \alpha}(x, \alpha) \quad \square$$

- If the assumptions of DP don't hold, then — in finite time dimensions — we can still solve using the Lagrange method or Kuhn-Tucker.

So the infinite dimension is more interesting here...

Example (Fisheries / Forest management).

We have $\forall t$ $\underbrace{s_{t+1} = A \cdot s_t(\bar{s} - s_t) + s_t - c_t}$,

assume $s_t \in [0, \bar{s}]$ for all t, $0 \leq c_t \leq s_t$, $\underbrace{0 \leq s_0 \leq \bar{s} \text{ given.}}$

We maximize the yield $u(c_t)$, discounted with $\beta \in (0,1)$

$$\max_{c_{t,\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} = V(s_0, 0) = \max_{c_0} \left\{ u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \right\} =$$

$\tau = t-1$

one could expand $\beta u(c_1) + \ldots$

$$= \max_{c_0} \left\{ u(c_0) + \beta \sum_{\tau=0}^{\infty} \beta^{\tau} u(c_{\tau+1}) \right\}$$

$$= \max_{c_0} \left\{ u(c_0) + \beta V(s_1, 1) \right\}.$$

Analogously,

$$V(s_t, t) = \max_{c_t} \left\{ u(c_t) + \beta V(s_{t+1}, t+1) \right\}.$$

Existence of the value fct will be shown later!

Solution steps:

1° In the {}:

$$\frac{\partial}{\partial c_t}: \quad u'(c_t) + \beta V'(s_{t+1}, t+1) \cdot \frac{\partial m_t}{\partial c_t} = 0$$

$$u'(c_t) + \beta V'(s_{t+1}, t+1) \cdot (-1) = 0$$

$$\underline{u'(c_t) = \beta V'(s_{t+1}, t+1)}$$

2° Use the envelope theorem

$$\frac{\partial}{\partial s_t}: \quad V'(s_t) = \cancel{\max_{c_t}} \left\{ \beta V'(s_{t+1}, t+1) \cdot \left(1 + A(\bar{s} - 2 s_t)\right) \right\}$$

3° Put together: $u'(c_t) = V'(s_t, t) \cdot \frac{1}{1 + A(\bar{s} - 2 s_t)}$

4° Shift by 1 period: $u'(c_{t+1}) = \frac{V'(s_{t+1}, t+1)}{1 + A(\bar{s} - 2 s_{t+1})}$

5° Insert again to get rid of V':

$$\boxed{u'(c_t) = \beta u'(c_{t+1}) \left(1 + A(\bar{s} - 2 s_{t+1})\right)} \leftarrow \text{Euler eq.}$$

E.g. if $u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta}$, $\theta > 0$, $\theta \neq 1$, (CRRA)

then $u'(c_t) = c_t^{-\theta}$, $\quad c_t^{-\theta} = \beta c_{t+1}^{-\theta} \left(1 + A(\bar{s} - 2 s_{t+1})\right)$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta\left(1 + A(\bar{s} - 2 s_{t+1})\right)$$

Assume that $(c_t, s_t) \xrightarrow{t \to \infty} \underbrace{(\tilde{c}, \tilde{s})}_{\text{STEADY STATE}}$ — we will show later that they will!

Then @SS, $c_{t+1} = c_t = \tilde{c}$, $s_{t+1} = s_t = \tilde{s}$.

$$\tilde{c} = A \tilde{s} (\bar{s} - \tilde{s}) = A \cdot \frac{1}{2}\left[\bar{s} - \frac{1-\beta}{A\beta}\right] \frac{1}{2}\left[\bar{s} + \frac{1-\beta}{A\beta}\right] = \frac{A}{4}\left(\bar{s}^2 - \left(\frac{1-\beta}{A\beta}\right)^2\right).$$

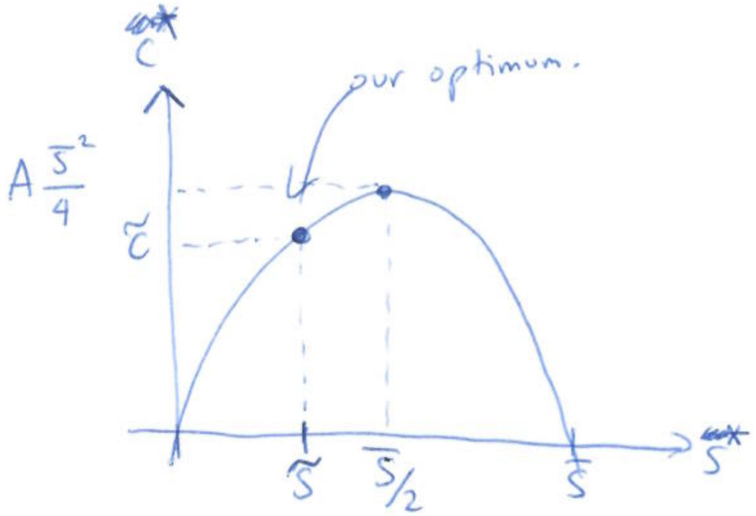
The steady state satisfies

$$1 = \beta\left(1 + A(\bar{s} - 2\tilde{s})\right)$$

$$\Rightarrow \frac{1}{\beta} - 1 = A(\bar{s} - 2\tilde{s})$$

$$\frac{1-\beta}{A\beta} = \bar{s} - 2\tilde{s}$$

$$\boxed{\tilde{s} = \frac{1}{2}\left[\bar{s} - \frac{1-\beta}{A\beta}\right] < \frac{1}{2}\bar{s}}$$



at s.s. $s_{t+1} = s_t$,

So $c^*(s^*) = A s^*(\bar{s} - s^*)$

Why? Because we discount the future. We consume more now, and so in the long run our yield $\boxed{\tilde{c}}$ is less than the highest long-run possible yield $\boxed{c^*}$

$$\frac{\partial \tilde{s}}{\partial \beta} = -\frac{1}{2}\left[\frac{-A\beta - A(1-\beta)}{(A\beta)^2}\right] = +\frac{1}{2}\,\frac{1}{A\beta^2} > 0$$

As $\beta \to 1_-$, $\tilde{s} \to \frac{1}{2}\bar{s}$, and thus $\tilde{c} \to \frac{A}{4}\bar{s}^2$.

More generally, discounting allows one to write

$$V(x_t, t) = \max_{u_t, \dots} \sum_{s=t}^{T-1} \underbrace{\alpha_s F_s(u_s, x_s)}_{f_s(u_s, x_s)}$$

Inserting to the Bellman eq.,

$$V(x_t, t) = \max_{u_t} \left\{ \alpha_t F_t(u_t, x_t) + V(x_{t+1}, t+1) \right\}$$

In current units $(/\alpha_t)$

$$V^c(x_t, t) = \max_{u_t} \left\{ F_t(u_t, x_t) + \underbrace{\frac{\alpha_{t+1}}{\alpha_t}}_{\beta_t \text{ - DISCOUNT FACTOR.}} V^c(x_{t+1}, t+1) \right\}$$

$$V^c(x_t, t) \overset{\text{def}}{:=} \frac{V(x_t, t)}{\alpha_t}$$

Infinite horizon stationary problem:

Ass. $F_s := F$, $m_s := m$; $C_s := C$; $\beta_s := \beta$

$$\frac{\alpha_{t+1}}{\alpha_t} \equiv const \Rightarrow \alpha_t = \alpha_0 \beta^t = \beta^t \quad (\text{w.l.o.g.})$$

The Bellman equation is:

$$V^c(x_t) = \max_{u_t} \{ F(x_t, u_t) + \beta V^c(x_{t+1}) \}$$

s.t. $x_{t+1} = m(u_t, x_t)$

$\forall t \geq 0$

- Policy fct $u^*(x_t)$ is time invariant $\boxed{!}$
- Value fct $V^c(x_t)$ — " —

Def. The Bellman operator $T: X \to X$, where X – space of fcts $v: \mathbb{R}^n \to \mathbb{R}$ is

$$Tv(x) = \max_u \{ F(x,u) + \beta v(y) \ \text{ s.t. } y = m(x,u), \ (x,u) \in C \}.$$

With the definition, the Bellman equation reads

$$V^c = TV^c.$$

Remark. Dynamic programming is also useful in problems with uncertainty: $x_{t+1} = \varphi(u_t, x_t, \underbrace{\varepsilon_{t+1}}_{\text{r.v.}})$;

$Pr(x_{t+1} \leq y \mid x_t, u_t) = G(y; x_t, u_t)$, for $y \in \Omega$

The Bellman eq. becomes

$$V(x_t) = \max_{u_t} \{ F(x_t, u_t) + \beta E_t V(x_{t+1}) \} =$$

$$= \max_{u_t} \{ F(x_t, u_t) + \beta \int_\Omega V(y) \, dG(y; x_t, u_t) \}$$

We know that if a value fct exists, it satisfies the Bellman eq. What about the converse?

It's also true!

Assume: $u_s \in \Gamma(x_s)$.

Theorem Let $V: \mathbb{R}^n \to \mathbb{R}$ solve

$$(*) \quad V(x) = \max_{u \in \Gamma(x)} \{ F(x,u) + \beta V(y) \text{ s.t. } y = m(x,u) \},$$

and let V satisfy $\lim_{n\to\infty} \beta^n V(x_n) = 0$ (boundedness) for any feasible $\{x_n\}$ from x_t onwards. Suppose that there exists $z^*_{t,\infty} = x_t \cup \{u_s^*, x_{s+1}^*\}_{s=t}^{\infty}$ where u_s^* solves

$$(**) \quad V(x_s) = \max_{u_s \in \Gamma(x_s)} \{ F(x_s, u_s) + \beta V(m(x_s, u_s)) \}$$

for each s and $x^*_{s+1} = m(x_s^*, u_s^*)$. Then V is the current value fct of the programming problem and $z^*_{t,\infty}$ solves

$$V(x_t) = \max_{u_{t,\infty}} \Big\{ \underbrace{\sum_{s=t}^{\infty} \beta^{s-t} F(x_s, u_s)}_{\text{Def } W(z_{t,\infty})} \text{ s.t. } x_{s+1} = m(x_s, u_s),\ x_t\text{-given},\ u_s \in \Gamma(x_s)\ \forall s \Big\}.$$

Proof: Let $z_{t,\infty} = x_t \cup \{u_s, x_{s+1} : s \geq t\}$ be an arbitrary admissible sequence, with x_t-given. Then by $(*)$

$$V(x_t) = \max_{u \in \Gamma(x_t)} \{ F(x_t, u_t) + \beta V(x_{t+1}) \} \geq$$

$$\geq F(x_t, u_t) + \beta V(x_{t+1}) \geq F(x_t, u_t) + \beta F(x_{t+1}, u_{t+1}) + \beta^2 V(x_{t+2})$$

$$\ldots \geq \sum_{s=t}^{T} \beta^{s-t} F(x_s, u_s) + \underbrace{\beta^T V(x_{t+T})}_{\xrightarrow{T\to\infty} 0}.$$

Given boundedness, $V(x_t) \geq \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, u_s) = W(z_{t,\infty})$.

Hence V is the upper bound for the value of the problem.

$\Rightarrow$

Moreover, $Z^*_{t,\infty} = x_t \cup \{u_s, x^*_{s+1}\}_{s \geq t}$ attains $V(x_t)$, by (***).

Finally, by definition,

$$V(x^*_t) = \max_{u_t \in \Gamma(x^*_t)} \{ F(u_t, x^*_t) + \beta V(m(x^*_t, u_t)) \} =$$

$$= F(u^*_t, x^*_t) + \beta V(\underbrace{m(x^*_t, u^*_t)}_{x^*_{t+1}}) =$$

$$= F(u^*_t, x^*_t) + \beta F(u^*_{t+1}, x^*_{t+1}) + \beta^2 V(x^*_{t+2}) = \ldots \overset{\text{by boundedness}}{=}$$

$$= \sum_{s=t}^{\infty} \beta^{s-t} F(u^*_s, x^*_s)$$ and thus $Z^*_{t,\infty}$ solves the program:

$$V(x^*_t) = W(Z^*_{t,\infty}). \quad \square$$

Example Job search model

— from other source $\square$

Example (Ramsey growth model)

full depreciation $\boxed{\delta = 1}$

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad k_{t+1} = f(k_t) - c_t$$

u, f – increasing & concave

$\beta \in (0,1)$

Bellman eq.

$$V(k_t) = \max_{c_t} \{ u(c_t) + \beta V(\underbrace{f(k_t) - c_t}_{k_{t+1}}) \}$$

$\boxed{a}$ Euler eq + dynamics

$\boxed{b}$ Characterize the value fct

$$\frac{\partial}{\partial c_t}: \quad u'(c_t) + \beta V'(k_{t+1})(-1) = 0$$

$$\beta V'(k_{t+1}) = u'(c_t)$$

$$\frac{\partial}{\partial k_t}: \quad V'(k_t) \underset{\substack{\uparrow \\ \text{env.} \\ \text{theorem}}}{=} \beta V'(k_{t+1}) f'(k_t)$$

Hence $u'(c_t) f'(k_t) = V'(k_t)$

$$\beta u'(c_{t+1}) f'(k_{t+1}) = \beta V'(k_{t+1})$$

$$\Rightarrow \underbrace{u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1})}_{\text{Euler eq.}}$$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta f'(k_{t+1})$$

Steady state (if exists)

(k^*, c^*) satisfies $\beta f'(k^*) = 1 \Rightarrow \boxed{k^* = (f^{-1})'\left(\frac{1}{\beta}\right)}$

$$\boxed{c^* = f(k^*) - k^*}$$

It can be shown that the optimal path $\{c_t, k_t\}_{t \geq 0}$ converges to (c^*, k^*).

<u>Dynamics</u> $\begin{cases} k_{t+1} = f(k_t) - c_t \\ c_{t+1} = \phi(k_t, c_t) \end{cases}$ as given <u>implicitly</u> by the Euler eq.

Let $\Phi(c_t, c_{t+1}, k_t) = u'(c_t) - \beta u'(c_{t+1}) f'(f(k_t) - c_t)$.

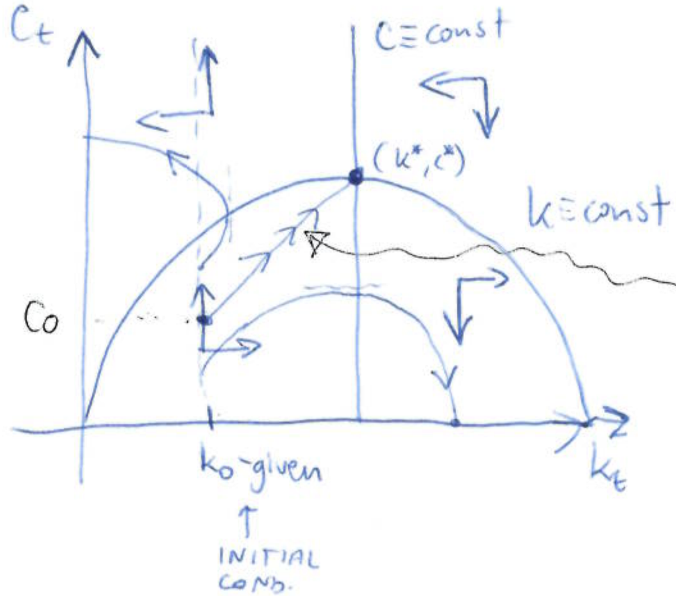
We have Euler eq. $\Leftrightarrow \Phi = 0$.

$$\frac{\partial \phi}{\partial k_t} = -\left.\frac{\frac{\partial \Phi}{\partial k_t}}{\frac{\partial \Phi}{\partial c_{t+1}}}\right|_{\Phi=0} = -\frac{+\overbrace{\beta u'(c_{t+1})}^{>0}\overbrace{f''(k_{t+1})}^{<0} \cdot \overbrace{f'(k_t)}^{>0}}{+\beta \underbrace{u''(c_{t+1})}_{<0} \underbrace{f'(k_{t+1})}_{>0}} < 0$$

IMPLICIT FCT THEOREM

$$\frac{\partial \phi}{\partial c_t} = - \frac{\frac{\partial \Phi}{\partial c_t}}{\frac{\partial \Phi}{\partial c_{t+1}}} = - \frac{\overbrace{u''(c_t)}^{<0} + \beta u'(c_{t+1}) \overbrace{f''(k_{t+1})}^{<0}}{-\beta u''(c_{t+1}) f'(k_{t+1})} > 0$$

Let's draw the phase diagram.



Isoclines

- $c \equiv const$
- $k \equiv const$

saddle path

Theorem If a sequence satisfies the F.O.Cs and the following transversality condition

$$\lim_{T \to \infty} \beta^T \cdot V'(k_T) k_T = 0$$

then it a solution to the optimization problem. !

Here, $V'(k_T) = u'(c_T) f'(k_T)$. If $k_S = 0$ for some s or $c_S = 0$ for some s then $V'(k_T) = +\infty$ and the transversality condition is violated. Thus the saddle path is the only path which can be the solution $\Rightarrow$ it is the solution.

Proposition The Bellman eq. has a unique continuous and bounded solution V. This value fct is strictly increasing & strictly concave. Furthermore $k_{t+1} = g(k_t)$ is a well-defined continuous function.

Proof First, for all $s \geq t$, $k_s \in [0, k_M]$ where k_M solves $f(k_M) = k_M$

and $c_s \in [0, k_s]$.

Boundedness. Let's write Bellman as

$$V(k_t) = \max_{k_{t+1}} \{ \underbrace{u(f(k_t) - k_{t+1})}_{u(c_t)} + \beta V(k_{t+1}) \}. \quad (*)$$

Fix $k_t^0 \in (0, k_M)$, let k_{t+1}^0 solve $(*)$.

Define $W(k_t) = u(f(k_t - k_{t+1}^0)) + \beta V(k_{t+1}^0)$ for $k_t \in B_\varepsilon(k_t^0)$. By continuity (and under border restrictions) $k_{t+1}^0 \in (0, f(k_t^0))$ — interior, and $\varepsilon > 0$ can be chosen small enough to guarantee $f(k_t) > k_{t+1}^0 \ \forall k_t \in B_\varepsilon(k_t^0)$, so

$\boxed{k_{t+1}^0}$ feasible $\forall k_t \in B_\varepsilon(k_t^0)$. Feasible, but not optimal:

$$W(k_t) \leq \max_{k_{t+1}} \{ u(f(k_t) - k_{t+1}) + \beta V(k_{t+1}) \} = V(k_t),$$

with $W(k_t^0) = V(k_t^0)$. (Existence & uniqueness).

- Differentiating, $V'(k_t) = \underbrace{u'(f(k_t) - k_{t+1})}_{>0} \underbrace{f'(k_t)}_{>0} > 0$.
- Differentiating wrt k_{t+1}, we get: $\underbrace{u'(f(k_t) - k_{t+1}) - \beta V'(k_{t+1})}_{= \Phi(k_t, k_{t+1})} = 0$

This yields implicit continuous $k_{t+1} = g(k_t)$,

$$\frac{\partial g}{\partial k_t} = \left. \frac{-\frac{\partial \Phi}{\partial k_t}}{\frac{\partial \Phi}{\partial k_{t+1}}} \right|_{\Phi = 0} \overset{\text{IMPLICIT FCT TH}}{=} - \frac{\overbrace{u''(c_t) \cdot f'(k_t)}^{<0}}{\underbrace{-u''(c_t)}_{<0} - \beta \underbrace{V''(k_{t+1})}_{<0}} > 0$$

Concavity of the value fct:

$$V''(k_t) = \underbrace{u''(c_t)}_{<0}(f'(k_t))^2 + u'(c_t)\cdot \underbrace{f''(k_t)}_{<0} \leq 0 \quad \square$$

~~But V needn't be twice differentiable~~ ??

Proposition The optimal capital sequence is monotonic.

- $g()$ is increasing. Hence if $k_1^* = g(k_0) \geq k_0$ then $k_2^* = g(k_1^*) \geq k_1^*$. True $\forall n$ by math. induction

Analogously, if $k_1^* \leq k_0$ then $\forall n$ the sequence is decreasing.

Proposition The sequence converges monotonically to k^*, from above if $k_0 > k^*$ and from below if $k_0 < k^*$.

Proof We know that V' is strictly decreasing.

Take k_t^* and $k_{t+1}^* = g(k_t^*)$. We know $\{k_t^*\}$ is monotonic. Hence $k_{t+1}^* \underset{<}{\geq} k_t^* \Leftrightarrow V'(k_{t+1}^*) \underset{>}{\leq} V'(k_t^*)$.

We know that: $V'(k_t^*) = u'(c_t^*) f'(k_t^*)$

$V'(k_{t+1}^*) = \frac{1}{\beta} u'(c_t^*)$.

Hence $k_{t+1}^* \underset{(<)}{>} k_t^* \Leftrightarrow \beta \cancel{u'(c_t^*)} f'(k_t^*) > \cancel{u'(c_t^*)}$

$\Leftrightarrow \beta f'(k_t^*) \underset{(<)}{>} 1$.

Since f' is decreasing and $\beta f'(k^*) = 1$,

$k_{t+1}^* \underset{(<)}{>} k_t^* \Leftrightarrow k_t < k^* \Leftrightarrow k_0 \underset{(>)}{<} k^*$.

Every monotone bounded sequence in $\mathbb{R}$ has a limit.

There is only one candidate for the limit:

$$k^* = \lim_{t\to\infty} k_{t+1}^* = \lim_{t\to\infty} g(k_t^*) = g\left(\lim_{t\to\infty} k_t^*\right) = g(k^*),$$

so it is our unique steady state. $\square$

Def. (Contraction)

Let (X,d) be a metric space, and $T: X \to X$.
We say that T is a contraction of modulus β if for some $\beta \in (0,1)$ we have:

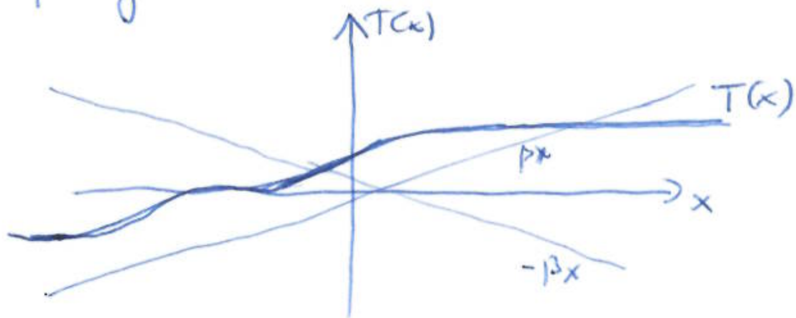
$$\forall x,y \in X \quad d(T(x),T(y)) \leq \beta d(x,y).$$

Ex. For $T:\mathbb{R}\to\mathbb{R}$ differentiable, we have

$$\forall x,y \in \mathbb{R} \quad |T(x)-T(y)| \leq \beta |x-y| \Leftrightarrow \frac{|T(x)-T(y)|}{|x-y|} \leq \beta .$$

Letting $y \to x$ we get the following characterization:

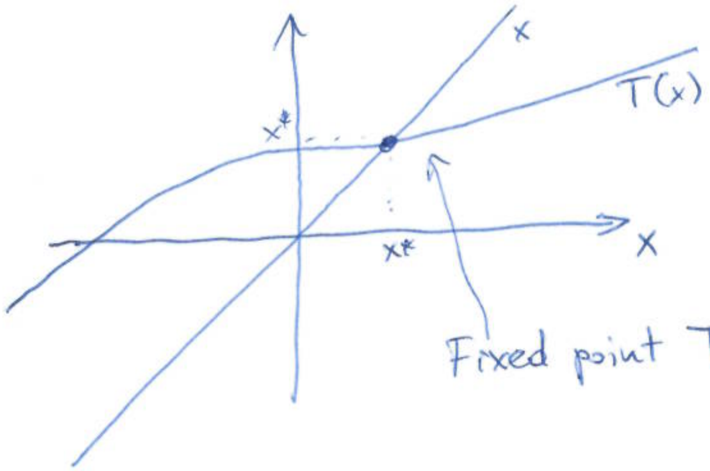
$$\forall x \in \mathbb{R} \quad \lim_{y\to x} \frac{|T(x)-T(y)|}{|x-y|} = |T'(x)| \leq \beta .$$

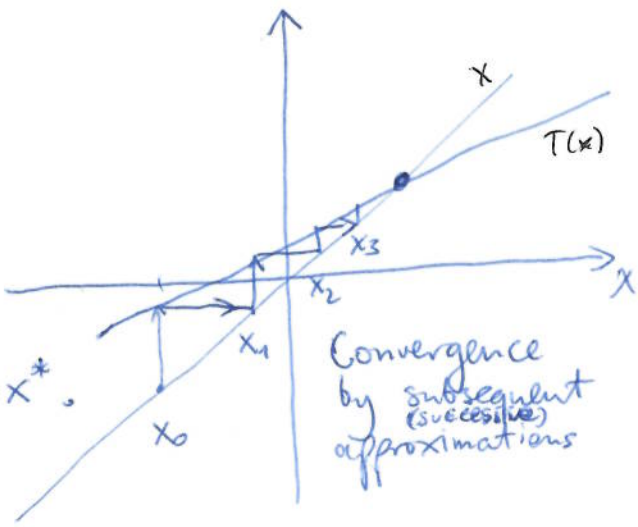


Fact. Every contraction is continuous.

Proof. We know that $d(T(x),T(y)) \leq \beta d(x,y) \quad \forall x,y \in X$.
Hence $\forall \varepsilon > 0 \ \exists \delta > 0 \ \ d(x,y) < \delta \Rightarrow d(T(x),T(y)) < \varepsilon$.
It suffices to take $\delta \leq \varepsilon/\beta$. Then $d(T(x),T(y)) \leq \beta d(x,y) < \varepsilon$ $\square$

Ex. Let $T: \mathbb{R} \to \mathbb{R}$





Theorem (Banach fixed point theorem a.k.a. contraction mapping theorem).

Let (X,d) be a complete metric space, and $T: X \to X$ be a contraction with modulus $\beta < 1$. Then

(a) T has exactly one fixed point, $T(x^*) = x^*$;

(b) the sequence $\{x_0, T(x_0), T(T(x_0)), \dots\}$ converges to $x^* \in X$ for every $x_0 \in X$.

Proof. Define $x_1 = T(x_0)$, and $x_{n+1} = T(x_n)$ $\forall n \in \mathbb{N}$.

- Owing to the fact that T is a contraction, we have

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq \beta d(x_n, x_{n-1}) \leq \beta^2 d(x_{n-1}, x_{n-2}) \leq \dots$$
$$\dots \leq \beta^n d(x_1, x_0).$$

- Consider ~~two~~ arbitrary terms in the sequence, x_m, x_n, with $m < n$.

Using the triangle inequality:

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + \dots + d(x_{m+1}, x_m) = \sum_{i=m}^{n-1} d(x_{i+1}, x_i) \leq$$
$$\leq \sum_{i=m}^{n-1} \beta^i d(x_1, x_0) = \beta^m d(x_1, x_0) \sum_{i=0}^{n-1-m} \beta^i \leq \frac{\beta^m}{1-\beta} d(x_1, x_0).$$

Hence as $m \to \infty$, $d(x_n, x_m) \to 0$ for any $n > m$. Thus the sequence is Cauchy. Since the space is complete, there exists $x^* \in X$ being the limit of $\{x_m\}$.

By continuity of T,

$$T(x^*) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus $T(x^*) = x^*$ and x^* is a fixed point of T.

To show uniqueness, proceed by contradiction. Let x', x'' be two different fixed points: $T(x') = x'$; $T(x'') = x''$.

Since T is a contraction,

$$d(x', x'') = d(T(x'), T(x'')) \leq \beta\, d(x', x''), \qquad \square$$

implying $1 \leq \beta$ — a contradiction. Hence $x' = x''$, and x^* is unique.

Remark It is sufficient that $T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ times}}$ is a contraction. Then T still has a unique fixed point.

Theorem (Continuous dependence of the fixed point on parameters)

Let (X, d) be the (complete) space of "variables", and (Ω, ρ) – the space of "parameters". Let $T: X \times \Omega \to X$, written as $T(x, \alpha)$, be:

- continuous w.r.t. α,
- $\forall \alpha \in \Omega$ $T(x, \alpha) := T_\alpha(x)$ a contraction w.r.t. x,

then $x^*(\alpha)$, $x^*: \Omega \to X$, i.e., the fixed point computed as a function of parameters, is continuous.

Proof. Let $\{\alpha_n\} \xrightarrow{n\to\infty} \alpha$. We would like to show that $x^*(\alpha_n) \xrightarrow{n\to\infty} x^*(\alpha)$.

By definition, $T_\alpha(x^*(\alpha)) = x^*(\alpha)$ for all $\alpha \in \Omega$.

We have $d(x^*(\alpha_n), x^*(\alpha)) = d(T_{\alpha_n}(x^*(\alpha_n)), T_\alpha(x^*(\alpha))) \leq$

$$\leq d(T_{\alpha_n}(x^*(\alpha_n)), T_{\alpha_n}(x^*(\alpha))) + d(T_{\alpha_n}(x^*(\alpha)), T_\alpha(x^*(\alpha))) \leq$$

$$\leq \beta\, d(x^*(\alpha_n), x^*(\alpha)) \cancel{\text{Hence we obtain}} + d(T_{\alpha_n}(x^*(\alpha)), T_\alpha(x^*(\alpha))).$$ Hence,

$$d\left(x^*(\alpha_n), x^*(\alpha)\right) \leq \frac{1}{1-\beta} d\left(T_{\alpha_n}(x^*(\alpha)), T_\alpha(x^*(\alpha))\right).$$

The right hand-side tends to 0 as $\alpha_n \xrightarrow{n\to\infty} \alpha$ due to continuity of T w.r.t. α. Hence $x^*(\alpha)$ is continuous. $\square$

Theorem (Blackwell) Sufficient conditions for a contraction.
Let $B(\mathbb{R}^n, \mathbb{R})$ be the set of bounded functions $f: \mathbb{R}^n \to \mathbb{R}$, with the sup norm $\|\cdot\|$: $\|f\| = \sup_{x\in\mathbb{R}^n} |f(x)|$.
If an operator $T: B(\mathbb{R}^n, \mathbb{R}) \longrightarrow B(\mathbb{R}^n, \mathbb{R})$ satisfies:

(a) MONOTONICITY: $\forall f,g \in B(\mathbb{R}^n, \mathbb{R})$ $\left(\forall x\in\mathbb{R}^n\ f(x) \leq g(x) \Rightarrow \forall x \in \mathbb{R}^n\ T(f(x)) \leq T(g(x))\right)$

(b) DISCOUNTING: $\exists \beta \in (0,1)$ $\forall f \in B(\mathbb{R}^n, \mathbb{R})$, $x\in\mathbb{R}^n$, $\alpha \geq 0$ we have $\underline{T(f(x)+\alpha) \leq T(f(x)) + \beta\alpha}$

then T is a contraction.

Proof. For any $f, g \in B(\mathbb{R}^n, \mathbb{R})$ we have

$$f = g + (f-g) \leq g + \|f-g\|$$

Hence: $T(f) \underset{\substack{\uparrow \\ \text{monotonicity}}}{\leq} T(g + \|f-g\|) \underset{\substack{\uparrow \\ \text{discounting}}}{\leq} T(g) + \beta\|f-g\|.$

And so $T(f) - T(g) \leq \beta\|f-g\|$.

Analogously, $g = f + (g-f) \leq f + \|f-g\|$, so
$T(g) \leq T(f) + \beta\|f-g\|$, so $T(f) - T(g) \geq -\beta\|f-g\|$.

Finally $\|T(f) - T(g)\| \leq \beta\|f-g\|$ for some $\beta\in(0,1)$, and thus T is a contraction. $\square$

Let us apply the Blackwell theorem to the existence of value fcts.

Recall the Bellman equation

$$V(x_t) = \max_{u_t \in \Gamma(x_t)} \{ F(x_t, u_t) + \beta V(x_{t+1}) \},$$

where $x_{t+1} = m(x_t, u_t)$.

V is defined as a fixed point of the operator T,
where

$$T(V(x)) = \max_{u \in \Gamma(x)} \{ F(x,u) + \beta V(m(x,u)) \}. \qquad (*)$$

If T is a contraction and $\Gamma(x)$ is compact, then a fixed point V exists and is unique.

Theorem (application of Blackwell & Weierstrass)

Let $F: \mathbb{R}^n \to \mathbb{R}$ be bounded and continuous.
Let $m: \mathbb{R}^n \to \mathbb{R}^k$ be continuous, and Γ be continuous w.r.t. x,
with $\Gamma(x)$ nonempty and compact for all x.
Then $T: C(X) \to C(X)$, where $X \subset \mathbb{R}^k$, is a contraction
and therefore has a unique fixed point $V \in C(X)$.
This fixed point is the value fct of the optimization problem.

Proof. Let $V \in C(X)$. The problem $(*)$ is well-defined because it is a problem of maximizing a continuous function on a compact set. By the Weierstrass theorem, a maximum exists, and thus $T(V(x))$ exists. Since F and V are continuous and bounded, $T(V)$ is also continuous and bounded.
Therefore $T: C(X) \to C(X)$. We will now show that T satisfies both Blackwell conditions — which is enough to prove that T is a contraction. As $C(X)$ is a complete metric space, we conclude that T has a unique fixed point V^*.

We have:

(a) monotonicity:

Let $w(x) \leq v(x)$ $\forall x \in X$. Then also $w(m(x,u)) \leq V(m(x,u))$

and so

$$T(V(x)) = \max_{u \in \Gamma(x)} \{ F(x,u) + \beta V(m(x,u)) \} \geq \max_{u \in \Gamma(x)} \{ F(x,u) + \beta w(m(x,u)) \}$$
$$= T(w(x))$$

(b) discounting:

$$T(V(x)+a) = \max_{u \in \Gamma(x)} \{ F(x,u) + \beta [V(m(x,u)) + a] \} =$$
$$= \max_{u \in \Gamma(x)} \{ F(x,u) + \beta V(m(x,u)) \} + \beta a = T(V(x)) + \beta a. \quad \square$$

Ex. (The Ramsey optimal growth model.)

Recall the Bellman eq.:

$$V(k_t) = \max_{c_t \in [0, f(k_t)]} \{ u(c_t) + \beta V(f(k_t) - c_t) \}, \text{ where } c_t \in [0, f(k_t)],$$
$$k_t \in [0, f(k_M)]$$
$$\forall t = 0,1,2,\ldots$$

! $$\boxed{T(V(k)) = \max_{c \in [0, f(k)]} \{ u(c) + \beta V(f(k) - c) \}}$$

- the "max" is well-defined due to Weierstrass.
- Blackwell conditions are verified:

(a) $\rightarrow$ let $w(k_t) \leq v(k_t)$ $\forall k_t \in [0, f(k_M)]$.

Then $T(w(k)) = \max_c \{ u(c) + \beta w(f(k) - c) \}$
$$\leq \max_c \{ u(c) + \beta v(f(k) - c) \} = T(v(k)).$$

(b) $\rightarrow$ $T(V(k)+a) = \max_c \{ u(c) + \beta [V(f(k) - c) + a] \} =$
$$= \max_c \{ u(c) + \beta V(f(k) - c) \} + \beta a = T(V(k)) + \beta a.$$

Example Job search model.

$\forall\, t = 0, 1, 2, \ldots$ there appears a wage offer w_t, drawn from a distribution $U[a,b]$: $w \sim U[a,b]$, $b > a > 0$.

Once in a job, a worker can get fired with probability $\lambda > 0$. The objective is to maximize the discounted sum of wages.

$$\max E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \cdot w_t \right\}.$$

We can write value function as follows:

$$V_0(w) = \max\{V_u(w); V_e(w)\}, \text{ where}$$

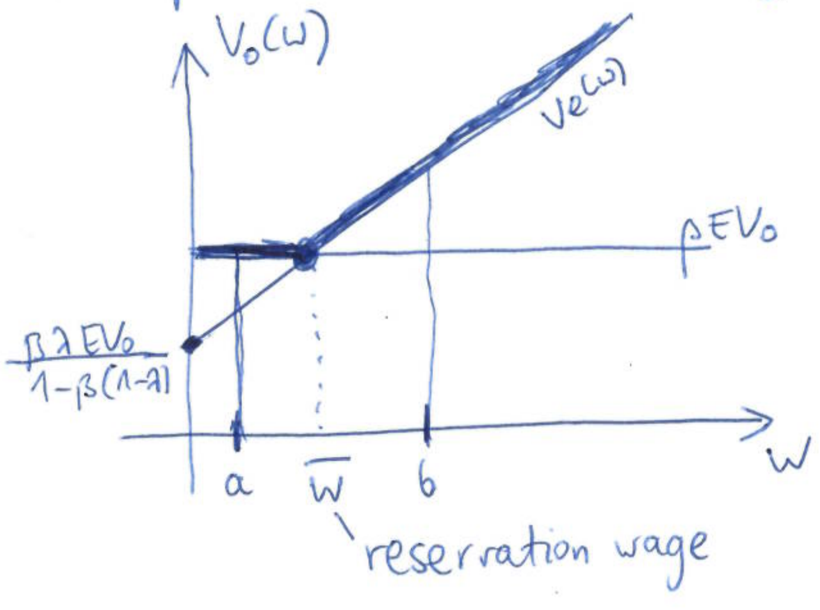
$$V_u(w) = \beta \cdot E V_0$$

$$V_e(w) = w + \beta\lambda \cdot E V_0 + \beta(1-\lambda) \cdot V_e(w).$$

Upon transformation,

$$V_e(w) = \frac{w + \beta\lambda E V_0}{1 - \beta(1-\lambda)}.$$

Which option to choose (optimally)?



Reservation wage:

$$V_u(\bar{w}) = V_e(\bar{w})$$

$$\beta EV_0 = \frac{\bar{w} + \beta\lambda EV_0}{1-\beta(1-\lambda)} \Rightarrow \boxed{\bar{w} = \beta(1-\beta)(1-\lambda)EV_0} \quad (*)$$

It remains to find EV_0.

$$EV_0 \stackrel{def}{=} \int_{\mathbb{R}} V_0(w)\, dF(w) = \int_a^b V_0(w)\cdot\frac{1}{b-a}\, dw =$$

$$= \frac{1}{b-a}\int_a^b V_0(w)\, dw = \frac{1}{b-a}\left[(b-a)\beta EV_0 + \int_{\bar{w}}^b \left(\frac{w+\beta\lambda EV_0}{1-\beta(1-\lambda)} - \beta EV_0\right) dw\right]$$

$$= \beta EV_0 + \frac{1}{b-a}\int_{\bar{w}}^b \left(\frac{w+\beta\lambda EV_0}{1-\beta(1-\lambda)} - \beta EV_0\right) dw$$

Denote $x := EV_0$, then

$$(1-\beta)x = \frac{1}{b-a}\left[\int_{\bar{w}}^b \left(\frac{w}{1-\beta(1-\lambda)}\right) dw + \int_{\bar{w}}^b \left(\frac{\beta\lambda}{1-\beta(1-\lambda)} - \beta\right) x\, dw\right] \Rightarrow$$

$$\ldots \Rightarrow \boxed{x = \frac{1}{1-\beta}\left[\frac{\frac{1}{2}(b^2-\bar{w}^2)}{b-a-\beta(1-\lambda)(\bar{w}-a)}\right]} \quad (**)$$

You may infer $\bar{w}$ and $x = EV_0$ from $(*)$ and $(**)$.

$\boxed{\text{Case } \lambda = 1}$ Then $\bar{w} = 0$ (we accept any positive wage)

$$V_0(w) = V_e(w)$$

$$V_e(w) = w + \beta EV_0 = w + \beta EV_e.$$

So, $EV_0 = EV_e = E(w + \beta EV_e) = Ew + \beta EV_e = \frac{a+b}{2} + \beta EV_e$

$$\Rightarrow EV_e = \frac{1}{1-\beta}\cdot\frac{a+b}{2} = \frac{a+b}{2}\left(1+\beta+\beta^2+\beta^3+\ldots\right)$$

Final exam "Optimization" 17H-19H. 4 december

QEM. Delay : 2H, no documents, no computers, no electronic devices, no cellphone!

Exercise 1

Consider 11 cities $A, B, C, D, B', C', D', B'', C'', D''$ and E. Some Cities are connected by roads, with the following distances :
$AB = 1, AC = 2, AD = 6, BC = 2, CD = 2, BB' = 5, CC' = 3, DD' = 1, B'C' = 1, C'D' = 1, B'B'' = 5, C'C'' = 5, D'D'' = 2, B''C'' = 1, C''D'' = 2, B''E = 1, D''E = 3, C''E = 2$.
This means, for example, you can go directly from A to B in 1 km. But B and C' (for example) are not directly connected.

For every city M, call $V(M)$ the distance of the shortest path from M to E. Compute, by a dynamic programming argument (backward induction), $V(M)$ for every point M. Find one shortest path from A to E.

(**Remark : Please**, draw a graph to represent the problem, each point of the graph being a city, an edge being a road; represent the distance between two cities by a number on the corresponding edge, and represent $V(M)$ for every M by a number on (or close to) M. But also explain the computation of $V(M)$).

Exercise 2

Let $0 < \alpha, 0 < \beta < 1$ and $k > 0$. Consider the following optimization problem

$$V(k) = \sup_{k_0=k, \forall t \in \mathbb{N}, 0 < k_{t+1} < k_t^\alpha} \sum_{t=0}^{+\infty} \beta^t ln(k_t^\alpha - k_{t+1}).$$

$k_{t+1} = k_t^\alpha - c_t$

$c_t = k_t^\alpha - k_{t+1}$

a) Recall quickly why $\sum_{t=0}^{+\infty} t.\beta^t$ converges.
b) For every $k > 0$, let $C(k) = \{(k_t)_{t\in\mathbb{N}} : k_0 = k, \forall t \in \mathbb{N}, 0 < k_{t+1} < k_t^\alpha\}$ the set of constraints. Prove that for every $(k_t)_{t\in\mathbb{N}} \in C(k)$, $\sum_{t=0}^{+\infty} \beta^t ln(k_t^\alpha - k_{t+1})$ is well defined.
c) For every function $f :]0, +\infty[\to \mathbb{R}$, define a new function $T(f)$ from $]0, +\infty[\to \mathbb{R}$ by : $\forall k > 0, T(f)(k) = \sup\{ln(k^\alpha - x) + \beta f(x), 0 < x < k^\alpha\}$. Prove that V satisfies the Bellman equation $T(V) = V$.
d) Recall the assumptions, in the course, which insures that the Bellman equation has a unique solution. Are these assumptions true here?

Exercise 2

(a) $\sum_{t=0}^{\infty} t\beta^t < \infty$. Intuitively, t grows linearly and thus is dominated by β^t which declines exponentially/geometrically.

$t < \left(\frac{\gamma+\beta}{\beta}\right)^t$

$t < \left(1+\frac{\gamma}{\beta}\right)^t$

Formally, $\forall \beta \in (0,1)\ \exists \gamma > 0\ \ \beta + \gamma < 1$.

For this γ, from some $\tilde{n} \in \mathbb{N}$ onwards, $\boxed{\beta^t t < (\gamma+\beta)^t}$ if $t > \tilde{n}$.

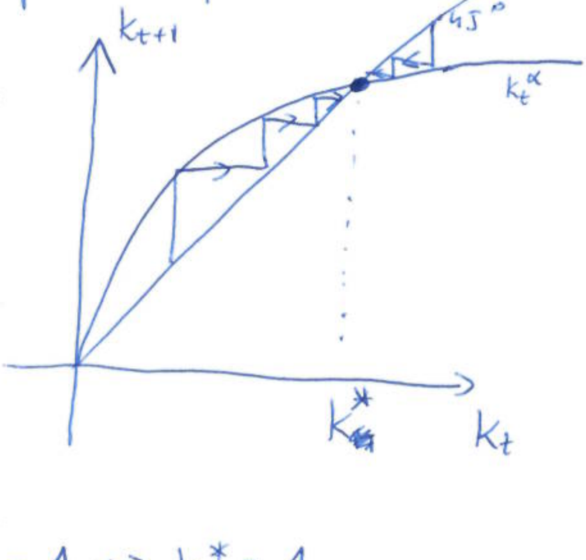
$$\text{Hence, } \sum_{t=0}^{\infty} t\beta^t = \sum_{t=0}^{\tilde{n}} t\beta^t + \sum_{t=\tilde{n}+1}^{\infty} t\beta^t < \underbrace{\sum_{t=0}^{\tilde{n}} t\beta^t}_{\text{finite sum}} + \underbrace{\sum_{t=\tilde{n}+1}^{\infty} (\beta+\gamma)^t}_{\text{finite}} < \infty.$$

Or, better, use Cauchy criterion,

$$q = \lim_{t\to\infty} \sqrt[t]{t\beta^t} = \beta \lim_{t\to\infty} \sqrt[t]{t} = \beta < 1.$$

(b) $C(k) = \{(k_t)_{t\in\mathbb{N}} : k_0 = k,\ \forall t \in \mathbb{N}\ \ 0 < k_{t+1} < k_t^\alpha\}$.

Let us consider the case of maximum possible capital accumulation (limiting case): $\forall t \geq 0\ \ k_{t+1} = k_t^\alpha$.



UPPER BOUND

- Hence, if $k_0 \in (0, k^*]$ then $\forall t\ \ k_t \leq k^*$.
- Alternatively, if $k_0 > k^*$ then $\forall t\ \ k_t \leq k_0$.
- We find that $k^* = (k^*)^\alpha \iff (k^*)^{1-\alpha} = 1 \iff \underline{k^* = 1}$

Then for every $(k_t)_{t\in\mathbb{N}} \in C(k)$, either

$$\sum_{t=0}^{\infty} \beta^t \ln(\underbrace{k_t^{\alpha} - k_{t+1}}_{c_t}) < \sum_{t=0}^{\infty} \beta^t \ln(k^*) = \ln(k^*)\frac{1}{1-\beta} < \infty \quad (k_0 \leq k^*)$$

or

$$\sum_{t=0}^{\infty} \beta^t \ln(\underbrace{k_t^{\alpha} - k_{t+1}}_{c_t}) < \sum_{t=0}^{\infty} \beta^t \ln(k_0^{\alpha}) = \ln(k_0^{\alpha})\frac{1}{1-\beta} < \infty \quad (k_0 > k^*)$$

LOWER BOUND (?)

- ~~$\forall t$ $k_0 \in (0, k^*]$ then $k_t \geq k_0$ $\forall t$.~~
- ~~$\forall t$ $k_0 > k^*$ then $k_t \geq k^*$ $\forall t$.~~

~~Hence~~

It seems that, although the supremum is well-defined here – the considered series is bounded above by a respective geometric series, it is not true in general, for every $(k_t)_{t\in\mathbb{N}} \in C(k)$.

- It would be true if we assumed (e.g.) that $c_t := k_t^{\alpha} - k_{t+1} \geq \tilde{c}$ $\forall t \geq 0$ – that there is a required minimum level of consumption in each period.

  Then, the lower bound would be

$$\sum_{t=0}^{\infty} \beta^t \ln c_t \geq \sum_{t=0}^{\infty} \beta^t \ln \tilde{c} = \frac{\ln \tilde{c}}{1-\beta} > -\infty .$$

- However, given logarithmic utility (implying that $c = 0 \Rightarrow \ln c = -\infty$) one could construct a case where the considered series diverges to $-\infty$.
- (Clearly not optimal, though.)

- Counterexample:

let $c_t = k_t^\alpha - k_{t+1} = \underline{\underline{k_0 e^{-\gamma^t}}}$, where $\gamma > \frac{1}{\beta} > 1$.

Note that $c_t \to 0$ very fast but $c_t > 0 \ \forall t$.

Also, $c_t < k_t^\alpha$ ($k_{t+1} > 0$) for all t. Hence, the example belongs to $C(k_0)$.

- We have

$$\sum_{t=0}^{\infty} \beta^t \ln(k_0 e^{-\gamma^t}) = \underbrace{\sum_{t=0}^{\infty} \beta^t \ln k_0}_{\frac{\ln k_0}{1-\beta}} + \sum_{t=0}^{\infty} \beta^t \ln(e^{-\gamma^t}) =$$

$$= \frac{\ln k_0}{1-\beta} - \sum_{t=0}^{\infty} \beta^t \cdot \gamma^t = \underline{\underline{-\infty}}$$

(c) We define $\forall k > 0$,

$$T(v)(k) = \sup_{x \in (0, k^\alpha)} \{ \ln(k^\alpha - x) + \beta v(x) \}.$$

The Bellman equation is then indeed $V(k) = T(V)(k) \ \forall k$, or $V = T(V)$.

- Let us write down the Bellman equation:

$$V(k) = \sup_{\{k_t\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t \ln(k_t^\alpha - k_{t+1}) \quad \text{s.t.} \quad \begin{array}{l} k_{t+1} \in (0, k_t^\alpha) \\ k_0 = k \end{array}$$

- The closure of the interval is $[0, k_t^\alpha]$, a compact set.
- The series is bounded above by a real number (i.e., convergent around the sup), and there exist $(k_t) \in C(k)$ for which the series indeed converges.

- the maximized function is <u>continuous</u>, because it is a composition and sum of continuous functions
- moreover, $\Gamma(k_t) = [0, k_t^{\alpha}]$ is a continuous correspondence (uhc & lhc)
- Therefore we may write

$$V(k) = \max_{\substack{k_{t+1} \in [0, k_t^{\alpha}] \\ k_0 = k}} \left\{ \sum_{t=0}^{\infty} \beta^t \ln\left(k_t^{\alpha} - k_{t+1}\right) \right\}$$

- The maximum is well-defined by <u>Weierstrass</u> theorem.

• ~~We may write:~~

$$\cancel{V(k) = \max_{k_{t+1} \in [0, k_t^{\alpha}]} \left\{ \ln\left(k_0^{\alpha} - k_1\right) \right\}}$$

• Observing that the maximized function satisfies the criteria of <u>stationary dynamic programming problems</u>,

- time separability
- geometric discounting
- time-invariant $\underbrace{\text{maximized } \cancel{\text{fn}} \text{ one-period}}_{\text{"utility fct"}}$ functions and equations of motion,

we write

$$V(k_t) = \max_{k_{t+1} \in [0, k_t^{\alpha}]} \left\{ \sum_{\tau=t}^{\infty} \beta^{\tau - t} \ln\left(k_{\tau}^{\alpha} - k_{\tau+1}\right) \right\}$$

Therefore

$$V(k_t) = \max_{k_{\tau+1} \in [0, k_{\tau}^{\alpha}]} \left\{ \ln\left(k_t^{\alpha} - k_{t+1}\right) + \beta \sum_{\tau=t+1}^{\infty} \beta^{\tau-(t+1)} \ln\left(k_{\tau}^{\alpha} - k_{\tau+1}\right) \right\}$$

$$\boxed{\text{BELLMAN PRINCIPLE OF OPTIMALITY}} \rightarrow \; = \max_{k_{t+1} \in [0, k_t^{\alpha}]} \left\{ \ln\left(k_t^{\alpha} - k_{t+1}\right) + \beta V(k_{t+1}) \right\}$$

→ <u>BELLMAN EQUATION</u>.

Hence, omitting time subscripts,

$$V(k) = \max_{x\in[0,k^\alpha]} \left\{ \ln(k^\alpha - x) + \beta V(x) \right\} = \sup_{x\in(0,k^\alpha)} \left\{ \ln(k^\alpha - x) + \beta V(x) \right\}.$$

(d) The Bellman equation has a <u>unique solution</u> if:

— the Bellman operator $T: X \to X$, defined on a complete metric space, is a contraction with a param. $\alpha \in (0,1)$.

- Here, $X = B(\mathbb{R}_+, \mathbb{R})$ — space of bounded, continuous functions $f: [0, +\infty) \to \mathbb{R}$.

This space, with a supremum norm, is a complete metric space.

- It <u>is a contraction</u> due to the fact that Blackwell conditions are verified.

(a) <u>monotonicity</u>:

Let $w(k) \leq v(k)$ $\forall k \geq 0$. Then

$$T(w(k)) = \sup_{x\in(0,k^\alpha)} \left\{ \ln(k^\alpha - x) + \beta w(x) \right\} \leq$$

$$\leq \sup_{x\in(0,k^\alpha)} \left\{ \ln(k^\alpha - x) + \beta v(x) \right\} = T(v(k)).$$

(b) <u>discounting</u>:

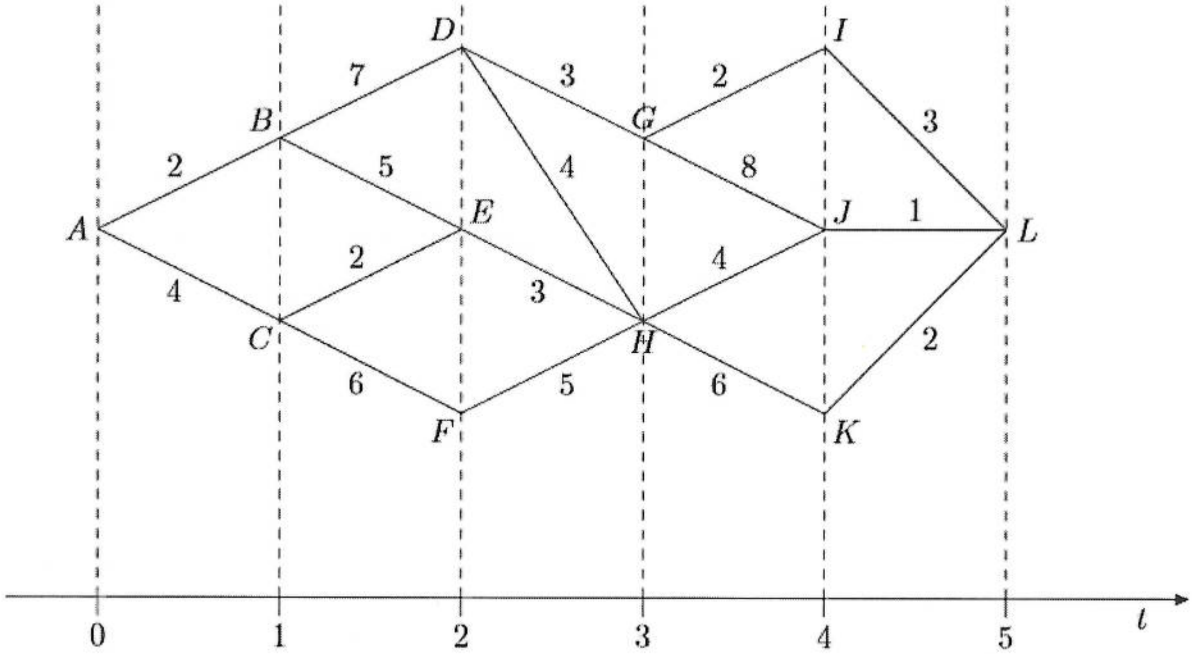
$$T(v(k) + \gamma) = \sup_{x\in(0,k^\alpha)} \left\{ \ln(k^\alpha - x) + \beta\left(v(x) + \gamma\right) \right\} =$$

$$= T(v(k)) + \beta\gamma, \quad \text{with } \beta \in (0,1).$$

- By Banach theorem (contraction mapping theorem), <u>the value function (fixed point of T) exists and is unique.</u>

Travel time

A1



- Find the route which guarantees minimum time of travel from A to L.
- Problem is discrete — finite number of available choices at each node.
- Let us compute value functions and policy functions, going backwards from L to A.

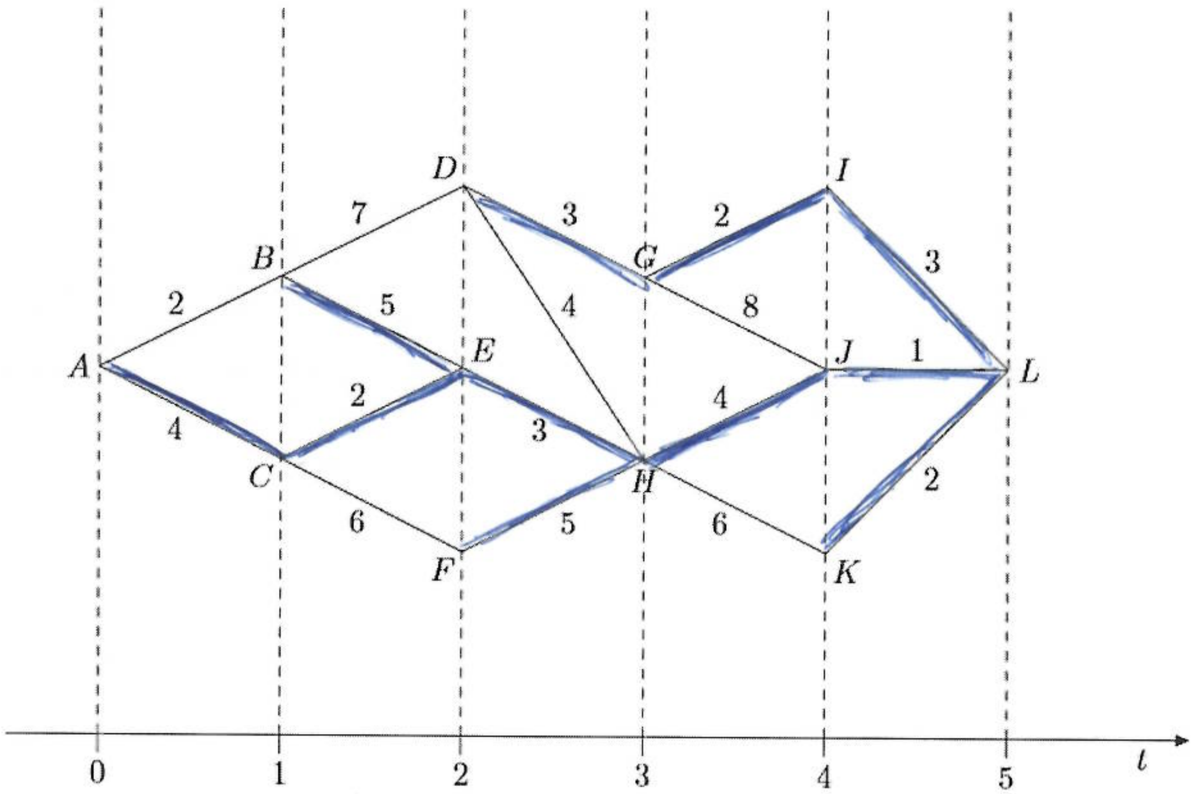
  Value fct: total time from the given node until L, assuming the quickest path

  Policy fct: which route to take from the given node in order to minimize the travel time to L.

  State variable: node [$s \in \{A, B, C, \dots, L\}$]

  Control variable: next node.

A2



$V(L) = 0$

$V(I) = 3 \qquad \varphi(I) = L$

$V(J) = 1 \qquad \varphi(J) = L$

$V(K) = 2 \qquad \varphi(K) = L$

$V(G) = \min\{5, 9\} = 5 \qquad \varphi(G) = I$

$V(H) = \min\{5, 8\} = 5 \qquad \varphi(H) = J$

$V(D) = \min\{8, 9\} = 8 \qquad \varphi(D) = G$

$V(E) = \min\{8\} = 8 \qquad \varphi(E) = H$

$V(F) = 10 \qquad \varphi(F) = H$

$V(B) = \min\{15, 13\} = 13 \qquad \varphi(B) = E$

$V(C) = \min\{10, 16\} = 10 \qquad \varphi(C) = E$

$V(A) = \min\{15, 14\} = 14, \ \varphi(A) = C$

Hence, the quickest path is $A \to C \to E \to H \to J \to L$.
Travel time = 14.

• More on the principle of backward induction. A3

Consider the problem: $\max_{\{c_t\}} \sum_{i=1}^{3} U_i$, where

$U_1 = \ln(c_1 c_2 c_3)$, $U_2 = \ln(c_2 c_3)$, $U_3 = \ln c_3$, subject to

$A_{t+1} = A_t - c_t$, A_1 – given, $A_t \geq 0$ for all $t = 1,2,3,4$.

Proceeding by backward induction, we have.

$t=3$ $\max_{c_3} \{\ln c_3\}$, s.t. $A_3 - c_3 \geq 0$

Clearly, one should consume all: $V_3(A_3) = \ln A_3$, $c_3(A_3) = A_3$.

$t=2$ Value function (Bellman eq.)

$$V_2(A_2) = \max_{c_2} \{\ln(c_2 c_3) + V_3(A_3)\} =$$
$$= \max_{c_2} \{\ln(c_2(A_2 - c_2)) + \ln(A_2 - c_2)\}$$

We have: $\frac{1}{c_2} + 2\frac{-1}{A_2 - c_2} = 0 \Rightarrow \frac{1}{c_2} = \frac{2}{A_2 - c_2} \Rightarrow 2c_2 = A_2 - c_2$
$\Downarrow$
$c_2 = \frac{1}{3} A_2$.

$$V_2(A_2) = \ln\left(\tfrac{1}{3}A_2\right) + 2\ln\left(\tfrac{2}{3}A_2\right); \quad c_2(A_2) = \tfrac{1}{3}A_2.$$

$t=1$ Value fct (Bellman eq.)

$$\boxed{\begin{matrix} A_2 = A_1 - c_1 \\ c_2 = \frac{1}{3}(A_1 - c_1) \end{matrix}}$$
$$c_3 = A_3 = A_2 - c_2 = \tfrac{2}{3}(A_1 - c_1)$$

$$V_1(A_1) = \max_{c_1} \{\ln(c_1 c_2 c_3) + V_2(A_2)\} =$$
$$= \max_{c_1} \Big\{\ln c_1 + \ln\left(\tfrac{1}{3}(A_1 - c_1)\right) + \ln\left(\tfrac{2}{3}(A_1 - c_1)\right) +$$
$$+ \ln\left(\tfrac{1}{3}(A_1 - c_1)\right) + 2\ln\left(\tfrac{2}{3}(A_1 - c_1)\right)\Big\}$$

A4

$$\frac{1}{c_1} + 2\,\frac{-1}{A_1-c_1} + 3\,\frac{-1}{A_1-c_1} = 0 \Rightarrow$$

$$\Rightarrow \frac{1}{c_1} = \frac{5}{A_1-c_1} \Rightarrow 5c_1 = A_1 - c_1 \Rightarrow \boxed{c_1(A_1) = \frac{1}{6}A_1}$$

Policy fct.

$$V_1(A_1) = \ln c_1 + 2\ln\frac{1}{3}(A_1-c_1) + 3\ln\frac{2}{3}(A_1-c_1) =$$

$$= \ln\frac{1}{6}A_1 + 2\ln\frac{5}{18}A_1 + 3\ln\frac{10}{18}A_1 =$$

$$= 6\ln A_1 + \underbrace{\ln\frac{1}{6} + 2\ln\frac{5}{18} + 3\ln\frac{5}{9}}_{\approx -6{,}12}$$

Let us solve the problem directly:

$$\max_{c_1,c_2,c_3}\left\{\underbrace{\ln(c_1c_2c_3) + \ln(c_2c_3)}_{F(c_1,c_2,c_3)} + \ln c_3\right\} \quad \text{s.t.} \begin{cases} A_1 - \text{given} \\ A_2 = A_1 - c_1 \\ A_3 = A_2 - c_2 \\ A_4 = A_3 - c_3 = 0 \end{cases}$$

See that $F(c_1, c_2, c_3) = \ln c_1 + 2\ln c_2 + 3\ln c_3$, and the restrictions boil down to: $\underbrace{c_1 + c_2 + c_3 = A_1}$.

The Lagrangean reads

$$\mathcal{L}(c_1,c_2,c_3) = \ln c_1 + 2\ln c_2 + 3\ln c_3 + \lambda(c_1+c_2+c_3-A_1)$$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda = 0 \\ \frac{\partial\mathcal{L}}{\partial c_2} = \frac{2}{c_2} + \lambda = 0 \\ \frac{\partial\mathcal{L}}{\partial c_3} = \frac{3}{c_3} + \lambda = 0 \\ c_1 + c_2 + c_3 = A_1 \end{cases} \Rightarrow \begin{cases} \frac{1}{c_1} = \frac{2}{c_2} \\ \frac{1}{c_1} = \frac{3}{c_3} \\ c_1 + c_2 + c_3 = A_1 \end{cases} \Rightarrow \begin{cases} 2c_1 = c_2 \\ 3c_1 = c_3 \\ 6c_1 = A_1 \end{cases}$$

(AT HOME: CHECK S.O.C.S)

$$\underbrace{c_1 = \frac{1}{6}A_1} \qquad \underbrace{c_2 = \frac{1}{3}A_1} \qquad \underbrace{c_3 = \frac{1}{2}A_1}$$

Maximum value equals to: $V_1^*(A_1) = \ln\frac{1}{6}A_1 + 2\ln\frac{1}{3}A_1 + 3\ln\frac{1}{2}A_1 =$

$$= 6\ln A_1 + \ln\frac{1}{6} + 2\ln\frac{1}{3} + 3\ln\frac{1}{2} \approx 6\ln A_1 - 6{,}07$$

A5

Why don't the two solutions coincide?

→ violation of the assumption of time separability of the objective function

→ preferences are time-inconsistent

→ dynamic programming yields sub-optimal allocations, (doesn't work!)

Let us now return to the fisheries / forest management problem.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad s_{t+1} = A s_t(\bar{s} - s_t) + s_t - c_t, \quad s_0 \in [0, \bar{s}] \text{ given}, \quad c_t \in [0, s_t] \text{ for all } t \geq 0.$$

Let us proceed to show existence & uniqueness of the value function of this problem.

~~1° For all $t \geq 0$,~~ ~~max~~

Bellman equation:

$$V_t(s_t) = \max_{c_t} \{ u(c_t) + \beta V_{t+1}(s_{t+1}) \}$$

$$\left(\text{since the problem is stationary: } V(s_t) = \max_{c_t} \{ u(c_t) + \beta V(s_{t+1}) \} \right).$$

1° For all $t \geq 0$, the problem is well-defined: (A6)

$\max_{c_t \in [0, s_t]} \{u(c_t) + \beta V(s_{t+1})\}$ consists in finding

a maximum of a continuous function on a compact set $[0, s_t]$. By Weierstrass theorem, a maximum is attained.

Proof of continuity of ~~$V(s_{t+1})$~~ V.

We know that $V(s_t) = \max_{\{c_\tau\}_{\tau \geq t}} \left\{ \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) \right\}$, given

the equation of motion. The equation of motion is obviously continuous, and u - continuous by assumption.

Hence $V(s_t) = \max_{\{s_\tau\}_{\tau \geq t+1}} \left\{ \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(As_\tau(\bar{s} - s_\tau) + s_\tau - s_{\tau+1}) \right\}$

is continuous (sums of continuous functions are continuous).

2° We can define: $T(V(s)) \stackrel{def}{:=} \max_{c \in [0, s]} \{u(c) + \beta V(As(\bar{s} - s) + s - c)\}$.

Note that the Bellman eq. now reads

$$V(s) = T(V(s)), \qquad \forall s \in [0, \bar{s}].$$

- Finding the value fct is equivalent to finding a fixed point of T.
- Banach's theorem (i.e., contraction mapping theorem) signifies that every contraction defined on a complete metric space has a unique fixed point.
- To do: – contraction? (Blackwell)
  – complete metric space?

(A7)

- Checking Blackwell's sufficient conditions:

  (a) <u>monotonicity:</u>

  Let $w(s) \leq v(s) \quad \forall s \in [0, \bar{s}]$.

  Then

$$T(w(s)) = \max_c \{u(c) + \beta w(As(\bar{s}-s)+s-c)\} \leq$$
$$\leq \max_c \{u(c) + \beta v(As(\bar{s}-s)+s-c)\} = T(v(s)).$$

  (b) <u>discounting</u>

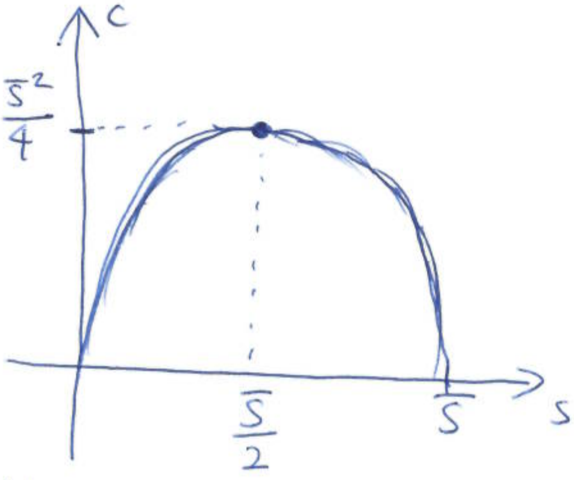
$$T(v(s)+\alpha) = \max_c \{u(c) + \beta[v(As(\bar{s}-s)+s-c)+\alpha]\} =$$
$$= \max_c \{u(c) + \beta V(As(\bar{s}-s)+s-c) + \beta\alpha\} =$$
$$= T(v(s)) + \beta\alpha.$$

  Hence T – contraction.

- Verification if $T: \underbrace{B(\mathbb{R}, \mathbb{R})}_{\text{set of bounded fcts}} \to B(\mathbb{R}, \mathbb{R})$ – which is a complete metric space.

$\to$ Does $V(s_t)$ converge for all $s_t \in [0, \bar{s}]$?

$$V(s_t) = \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) \leq$$
$$\leq \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(\cancel{A\tfrac{\bar{s}^2}{4}}\,\bar{s}) =$$
$$= u(\bar{s}) \sum_{s=0}^{\infty} \beta^s = \frac{u(\bar{s})}{1-\beta} < \infty.$$



<u>Yes</u>, and thus V – bounded.

- Application of Blackwell & Banach theorems

$\Rightarrow$ $\underbrace{\text{V exists and is unique.}}$ [It is also easily verified that it is continuous and differentiable]